

## COLE'S PROBLEM

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Consider the problem of finding a local extremal value of  $x_1(0)$  given the constraint that  $(x_1(t), x_2(t))$  be a periodic solution of the non-autonomous dynamical system

$$\dot{x}_1 = x_2, \dot{x}_2 = \cos(t + \phi) - x_2 - kx_1$$

for which  $x_2(0) = 0$ . It is straightforward to show that such periodic solutions are given by

$$x_1(t) = \frac{(k-1)\cos(t+\phi) + \sin(t+\phi)}{1+(k-1)^2}, \quad x_2(t) = -\frac{(k-1)\sin(t+\phi) - \cos(t+\phi)}{1+(k-1)^2}$$

provided that

$$\sin \phi = \frac{1}{\sqrt{1+(k-1)^2}}, \quad \cos \phi = \frac{k-1}{\sqrt{1+(k-1)^2}}$$

i.e.,

$$x_1(t) = \frac{\cos t}{\sqrt{1+(k-1)^2}}, \quad x_2(t) = -\frac{\sin t}{\sqrt{1+(k-1)^2}}$$

Since  $x_1(0) = 1/\sqrt{1+(k-1)^2}$ , it follows that a local extremum under variations in  $k$  occurs for  $k = 1$ , in which case  $\phi = \pi/2$ .

According to the theory of adjoints, we should be able to obtain this result by considering variations of the Lagrangian

$$\begin{aligned} L = & \mu_1 + \int_0^{2\pi} \lambda_1(t) (\dot{x}_1(t) - x_2(t)) dt + \int_0^{2\pi} \lambda_2(t) (\dot{x}_2(t) + x_2(t) + kx_1(t) - \cos(t + \phi)) dt \\ & + \lambda_3 (x_1(0) - x_1(2\pi)) + \lambda_4 (x_2(0) - x_2(2\pi)) + \lambda_5 x_2(0) + \eta (x_1(0) - \mu_1) \end{aligned}$$

with respect to all unknowns and Lagrange multipliers. In particular, we obtain the adjoint equations

$$\begin{aligned} -\dot{\lambda}_1 + k\lambda_2 &= 0, \quad -\dot{\lambda}_2 + \lambda_2 - \lambda_1 = 0, \quad \lambda_1(2\pi) - \lambda_3 = 0, \\ -\lambda_1(0) + \lambda_3 + \eta &= 0, \quad \lambda_2(2\pi) - \lambda_4 = 0, \quad -\lambda_2(0) + \lambda_4 + \lambda_5 = 0 \\ \int_0^{2\pi} \lambda_2(t)x_1(t)dt &= 0, \quad \int_0^{2\pi} \lambda_2(t)\sin(t+\phi)dt = 0, \quad 1 - \eta = 0 \end{aligned}$$

The differential equations together with the initial conditions yield

$$\lambda_1(t) = \frac{e^{t/2}}{\sqrt{1-4k}} \left( \sqrt{1-4k}(\eta + \lambda_3) \cosh \frac{\sqrt{1-4k}t}{2} - (\eta + \lambda_3 - 2k(\lambda_4 + \lambda_5)) \sinh \frac{\sqrt{1-4k}t}{2} \right)$$

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and

$$\lambda_2(t) = \frac{e^{t/2}}{\sqrt{1-4k}} \left( \sqrt{1-4k}(\lambda_4 + \lambda_5) \cosh \frac{\sqrt{1-4k}t}{2} + (-2\eta - 2\lambda_3 + \lambda_4 + \lambda_5) \sinh \frac{\sqrt{1-4k}t}{2} \right)$$

Substitution of  $t = 2\pi$  into the remaining boundary conditions then implies that

$$\lambda_3 = \frac{e^\pi(1-4k)\eta - (1-4k)\eta \cosh \sqrt{1-4k}\pi + \sqrt{1-4k}(\eta - 2k\lambda_5) \sinh \sqrt{1-4k}\pi}{2(1-4k)(\cosh \sqrt{1-4k}\pi - \cosh \pi)}$$

and

$$\lambda_4 = \frac{e^\pi(1-4k)\lambda_5 - \lambda_5(1-4k) \cosh \sqrt{1-4k}\pi + \sqrt{1-4k}(2\eta - \lambda_5) \sinh \sqrt{1-4k}\pi}{2(1-4k)(\cosh \sqrt{1-4k}\pi - \cosh \pi)}$$

Substitution into the first integral condition yields

$$\frac{(1-k)\eta - \lambda_5}{(2 + (k-2)k)^{3/2}} = 0 \Rightarrow \lambda_5 = \eta(1-k)$$

and, after substitution of this value of  $\lambda_5$  into the second integral condition, we obtain

$$\eta \cos \phi = 0 \Rightarrow \eta = 0 \text{ or } \phi = \pi/2$$

So, if  $k \neq 1$ , then  $\phi \neq \pi/2$  and, therefore,  $\eta$  must equal 0 (it cannot satisfy the necessary condition), in which case  $\lambda_5 = 0$ , in which case  $\lambda_3$  and  $\lambda_4$  both equal zero, in which case the functions  $\lambda_1$  and  $\lambda_2$  both equal 0 for all  $t$ . On the other hand, if  $k = 1$ , then  $\phi = \pi/2$  and  $\eta$  can equal 1, in which case  $\lambda_5 = 0$  and

$$\lambda_3 = \frac{3e^\pi - 3\cos \sqrt{3}\pi + \sqrt{3}\sin \sqrt{3}\pi}{6(\cos \sqrt{3}\pi - \cos \pi)}, \lambda_4 = \frac{\sin \sqrt{3}\pi}{\sqrt{3}(\cos \sqrt{3}\pi - \cos \pi)}$$

in which case  $\lambda_1$  and  $\lambda_2$  are both nonzero functions of  $t$ .